1.

M 250.
Burger

I. Rational Roots Theorem:

Given a polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = f(x)$ with 'integer coefficients', ie. $a_i \in \mathbb{Z}$.

Then the 'rational roots' (If they exist) are of the form $\frac{p}{q}$ where $p \mid a_0$ and $q \mid a_n$.

<u>pf.</u>

Let $\frac{p}{q}$ be a rational root of $f(x)$ in reduced fraction form.

then $f\left(\frac{p}{q}\right) = a_n \left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$

Trick!: Multiply Both sides by q^{n-1}.

$$\rightsquigarrow \quad a_n \frac{p^n}{q^n} \cdot q^{n-1} + a_{n-1} \cancel{\frac{p^{n-1}}{q^{n-1}} \cdot q^{n-1}} + \cdots + \frac{a_1 p}{q} \cdot q^{n-1} + a_0 q^{n-1} = 0 \cdot q^{n-1}$$

$$\rightsquigarrow \quad a_n \frac{p^n}{q} + \underbrace{a_{n-1} p^{n-1} + \cdots + \text{etc.}}_{\text{integers multiplying integers ie. No fractions.}} = 0$$

$q \nmid p$ since $\frac{p}{q}$ is reduced fraction.

$\therefore$ $q \mid a_n$ since we must get an integer here to make 0.

2. Next trick, multiply both sides by $\frac{q^n}{p}$ to get:

$$a_n \frac{p^n}{q^n} \cdot \frac{q^n}{p} + a_{n-1} \frac{p^{n-1}}{q^{n-1}} \cdot \frac{q^n}{p} + \cdots + a_1 \cancel{\frac{p}{q}} \cdot \frac{q^n}{\cancel{p}} + a_0 \cdot \frac{q^n}{p} = 0$$

$$\leadsto \underbrace{a_n p^{n-1} + a_{n-1} p^{n-2} \cdot q + \cdots + a_1 q}_{\text{All integers!}} + a_0 \cdot \frac{q^n}{p} = 0 \quad \frac{q^n}{p}$$

$\therefore$ For the same reason as the previous case $p \mid a_0$. For ex. $3 - \frac{7}{2}$ can never be zero if the fractions are reduced. ■

II. The Factor Theorem.

r is a root of $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ iff $x - r$ is a factor of $f(x)$.

pf. $(\leftarrow)$ If $(x-r) \mid f(x)$ then $f(x) = (x-r) \cdot \underbrace{q(x)}_{\text{some polynomial.}}$ $\therefore$ $f(r) = (r-r) \cdot q(r) = 0 \cdot q(r) = 0$. $\therefore r$ is a root.

3. ($\longrightarrow$) Suppose r is a root of $f(x)$, i.e.

$f(r) = 0$.

We haven't Discussed the Euclidean Algorithm for polynomials 'yet,' but by that Algorithm, we can divide $f(x)$ by $x-r$ such that:

$$f(x) = (x-r) \cdot q(x) + r(x) \quad \text{where}$$

$$\deg(r(x)) < \underset{\underset{1}{\shortparallel}}{\deg(x-r)}$$ in other

words, when dividing any polynomial by $x-r$, the remainder is a '**constant**'.

So let $r(x) = c$ in this case:

$$f(x) = (x-r)\,q(x) + c.$$

then by hypothesis:

$$f(r) = 0 = 0 \cdot q(x) + c$$

$$\longrightarrow \quad 0 = 0 + c \longrightarrow c = 0.$$

$$\therefore \quad f(x) = (x-r)\,q(x) \longrightarrow x-r \mid f(x).$$

∎

4. III. The Conjugate Complex and Irrational Roots theorems

All result as a consequence of the Quadratic Formula:

$ax^2+bx+c=0 \rightsquigarrow$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

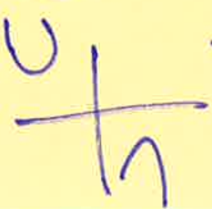
recall:

$\Delta = b^2-4ac$ 'the Discriminant'

Case 1. $\Delta = 0 \rightsquigarrow x = \frac{-b}{2a}$ is only root, i.e. a multiplicity two root.

Case 2 $\Delta < 0 \rightsquigarrow \sqrt{\Delta}$ is a complex #.

$\therefore \; x = \frac{-b + \text{complex \#}}{2a}$ ($z = ti$) and $x = \frac{-b - \text{complex \#}}{2a}$ ($z = ti$).



$\frac{-b}{2a} + \frac{z}{2a}$ $\quad$ $\frac{-b}{2a} - \frac{z}{2a}$ $\quad$ complex conjugates!

$\frac{-b}{2a} + \frac{ti}{2a}$ $\quad$ $\frac{-b}{2a} - \frac{ti}{2a}$

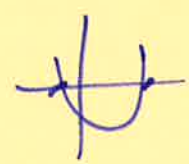
<u>Case 3.</u> $\Delta > 0$ Two real roots.

<u>3i</u> If Δ is a perfect square then we have two 'rational' roots.

since $\sqrt{\Delta}$ will be an integer. positive actually.

Let $\sqrt{\Delta} = t$

then $\dfrac{-b+t}{2a}$ and $\dfrac{-b-t}{2a}$ will be roots.



<u>3ii</u> If Δ is not a perfect square then $\sqrt{\Delta}$ will be an irrational #. (You should know how to prove $\sqrt{p}$ is irrational when p is prime.)

So lets just say $\Delta = 2$ for example. then one will obtain two conjugate irrational roots: $\dfrac{-b+\sqrt{2}}{2a}$ $\dfrac{-b-\sqrt{2}}{2a}$

ie.. $-b/2a + \dfrac{\sqrt{2}}{2a}$ $\quad -\dfrac{b}{2a} - \sqrt{2}/2a$

Lecture example =

1. A polynomial has roots -3, $\sqrt{2}$ and $5i$. what is the smallest degree it can have?

Preview of coming attractions: The answer 'could' be 3 if the polynomial ring has its scalars coming from $\mathbb{C}$.

i.e. $p(z) = (z+3)(z-\sqrt{2})(z-5i)$ is a polynomial 'over' $\mathbb{C}$ with only these roots.

But for polynomials 'over' ~~the reals~~. ~~$\mathbb{R}$~~. $\mathbb{Z}$.

the answer is 5.

For example: $p(x) = (x+3)(x-\sqrt{2})(x+\sqrt{2})(x-5i)(x+5i)$

Q. Is this the only one?

A. No. one could ~~add~~ mult. more factors, in or multiply by a constant. $k \cdot p(x)$ has same roots.

ex. 2 Find all the roots of $p(x) = x^3 + 2x^2 + 3x + 6$ and sketch a graph.

soln. By RRT, possible rational roots are $\frac{\pm 6}{1}, \frac{\pm 3}{1}, \frac{\pm 2}{1}, \frac{\pm 1}{1}$.

obviously we want to try a negative candidate.

$$p(-1) = -1 + 2 - 3 + 6 = 4 \neq 0.$$

$$p(-2) = -8 + 8 - 6 + 6 = 0.$$

Since $\deg p(x) = 3$ this is all we need since by the Factor theorem. $x+2 \mid p(x)$, which will give a deg 2 left that we could use the Q-Formula on.

(Introduce

Base. x	x^3	x^2	x	1
	1	2	3	6

9.

$$
\begin{array}{r|rrrr}
 & 1 & 0 & 3 & . \\
\hline
12_x & 1 & 2 & 3 & 6_x \\
 & -1 & 2 & \downarrow & \downarrow \\
\hline
 & & 0 & 3 & 6 \\
 & & & 3 & 6 \\
\hline
 & & & & 0.
\end{array}
$$

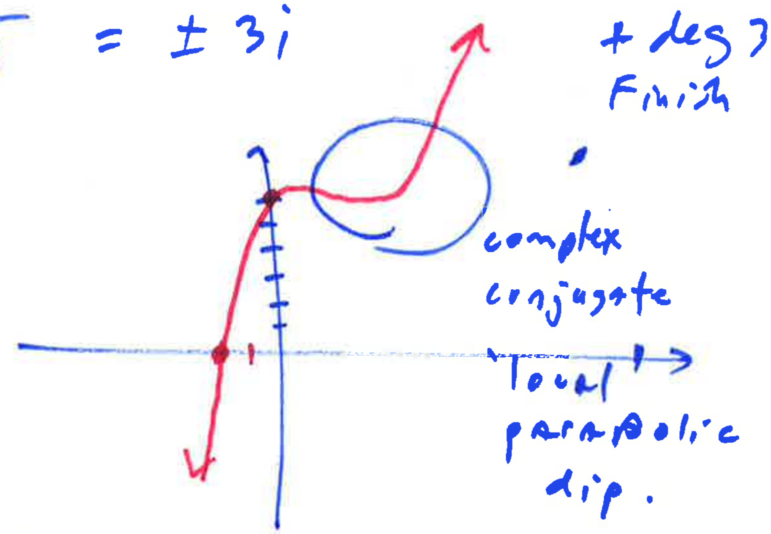
$\therefore \quad f(x) = (x+2)(x^2+3)$

$\rightsquigarrow \quad x^2+3 = 0$

$x^2 = -3$

$x = \pm\sqrt{-3} = \pm 3i$

$p(0) = 6$



Start
+ deg 3

'Around roots polynomials behave like the degree of multiplicity of the root.'

9.
ex 3. Find all the roots of

$$f(x) = 2x^4 + 9x^3 + 11x^2 + 9x + 9$$

given that $\sqrt{-1} = i$ is a root + sketch Graph.

solution.

By complex conj. root theorem. $-i$ is also a root $\therefore$ $(x-i)(x+i) = x^2+1 \mid f(x)$.

$$\begin{array}{r|rrrrr} & & & 2 & 9 & 9 \\ & x^4 & x^3 & x^2 & x & 1 \\ 101_x & 2 & 9 & (11) & 9 & 9 \\ & -(2 & 0 & 2) & \downarrow & \\ & & 9 & (9) & 9 & \\ & & -(9 & 0 & 9) & \cancel{x} \\ & & & 9 & 0 & 9 \\ & & & 9 & 0 & 9 \\ & & & & 0 & \end{array}$$

$$\therefore \quad f(x) = (x^2+1)(2x^2+9x+9).$$

For $2x^2+9x+9$ deg 2 $\therefore$ Q-Formula.

$$x = \frac{-9 \pm \sqrt{81-4(2)(9)}}{2(2)} = \frac{-9 \pm \sqrt{9}}{4} = \frac{-9 \pm 3}{4}$$

$$\frac{-9-3}{4} = \frac{-12}{4} = -3 \qquad \frac{-9+3}{4} = -\frac{6}{4} = -\frac{3}{2}.$$

10.

notice if we write all the deg 1 factors out:

$$(x^2+1)(x+3)(x+3/2)$$ it would not multiply out to $f(x)$ have a leading term of $2x^4$, and poly. would not have integer coefficients throughout.

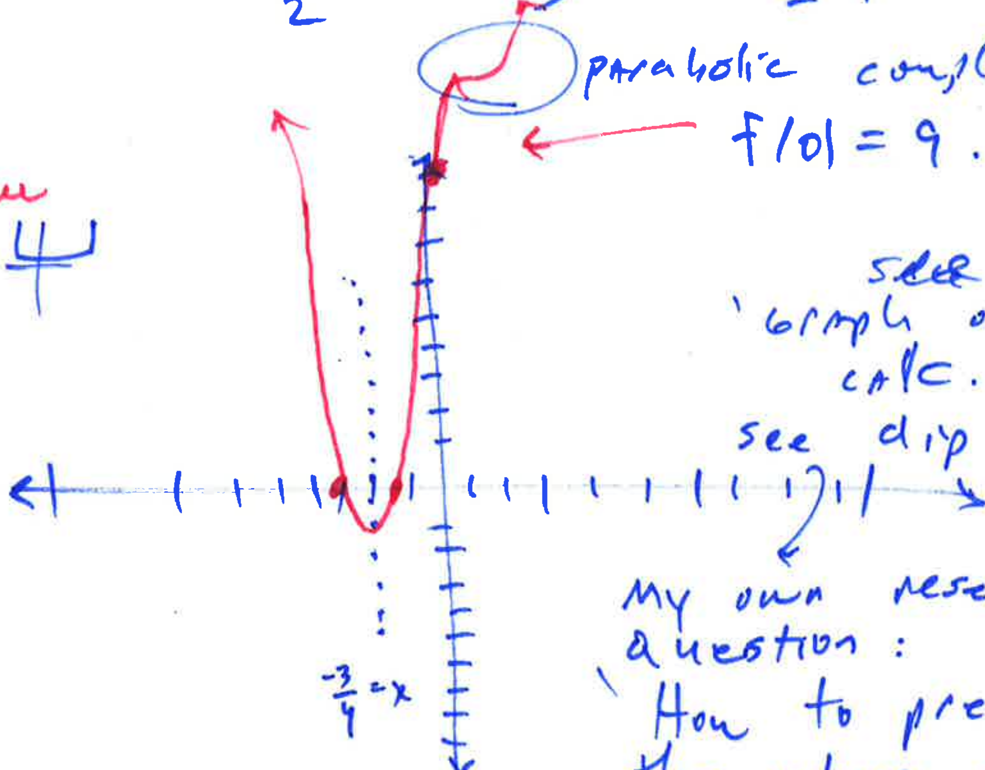
But $2(x+\frac{3}{2}) = 2x+3$

$$\therefore \quad f(x) = (2x+3)\cdot(x+3)\cdot(x^2+1)$$

$-\frac{3}{2}$ -3 $\pm i$

Sketch.

pos. even degree has basic shape



parabolic complex dip is somewhere.

$f(0) = 9$.

'see graph on calc. to see dip.'

My own research question:
'How to predict the nature of these dips?'

Day 1 end. ☺